References

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[2]Baruch, M. and Riff, R., "Hamilton Principle, Hamilton's Law, 6^n Correct Formulations," *AIAA Journal*, Vol. 20, May 1982, pp. 687-692.

[3]Fried, I., "Finite Element Analysis of Time Dependent Phenomena," *AIAA Journal*, Vol. 7, June 1969, pp. 1170-1173.

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Reply by Authors to M. Borri, M. Lanz, and P. Mantegazza

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WE would like to thank Professor Borri, Lanz, and Mantegazza for their interest in our paper[1] and for sharing with us the opinion that the extension of the finite element to the time domain is well motivated. It seems, however, that the authors of the Comment missed the main goal of the paper,[1] which was to introduce a new high-precision time finite element analogous to the standard finite element algorithms for which cases it may be needed.

The authors[1-3] (see also Refs. 4 and 5) are still convinced that the easiest and most consistent way to impose all of the initial dynamic conditions and to produce more accurate algorithms is through the third or higher order Hermitian polynomials. It appears,[2-3] and was pointed out in the paper,[1] that the so-called "straightforward way" in the Comment, which was introduced in Ref. 4, caused the time finite element algorithm to become unconditionally unstable.[2] On the contrary, the so-called "artificial and tricky" formulation of the paper,[1,6] succeeded in producing a stable converged time finite algorithm of sixth-order accuracy.

Moreover, in spite of the same stability characteristic, the step-by-step version of the algorithm presented in the Appendix of Ref. 1 possesses a much greater accuracy (by two orders) than the one presented in Ref. 5, which is based on the algorithm of Ref. 4. Furthermore, in spite of the same kind of modification (see Ref. 7) for unconditional stability, the algorithm presented in the Appendix of Ref. 1 is of fifth-order accuracy, while the one of Ref. 5 is of third-order accuracy due to the difference in the basic algorithms.

Of course, one can choose to exploit the power of time finite element discretization of Hamilton's Law to reproduce any already existing step-by-step integration methods. Clearly Eqs. (4) of the Comment are nothing else than the well known Newmark linear acceleration method of third-order accuracy.

In fact, the first author of this Reply had previously obtained[3] the same result as in the Comment and has introduced a general formulation of this kind of step-by-step method by applying time finite element discretization of Hamilton's Law. Similar results had been obtained earlier by Zienkiewicz[8] through weighted residuals methods.

The general formulation[3] is simply supported by approximating the displacement q and the variation S as follows,

$$q(t) = (1-\tau)q_{i-1} + \tau q_i \tag{1}$$

$$S(t) = G_i(\tau)S_{i-1} + G_2(\tau)S_i \tag{2}$$

where $\tau = t/\Delta t_i$. By substitution of Eqs. (1) and (2) into Eq. (4) of Ref. 1 (Hamilton's Law), one obtains:

$$\begin{bmatrix} -M+\gamma_1\Delta t_i C+(\beta_1-\gamma_1)\Delta t_i^2 K, & M-\gamma_1\Delta t_i C-\beta_1\Delta t_i^2 K \\ M-\gamma_2\Delta t_i C-(\beta_2-\gamma_2)\Delta t_i^2 K, & -M+\gamma_2\Delta t_i C+\beta_2\Delta t_i^2 K \end{bmatrix} \times \begin{Bmatrix} q_{i-1} \\ q_i \end{Bmatrix} = \Delta t_i^2 \begin{bmatrix} \beta_1-\gamma_1, & -\beta_1 \\ -\beta_2+\gamma_2, & \beta_2 \end{bmatrix} \begin{Bmatrix} f_{i-1} \\ f_i \end{Bmatrix} \tag{3}$$

where,

$$\beta_\alpha = \int_0^1 G_\alpha \tau \mathrm{d}\tau \Big/ \int_0^1 \dot{G}_\alpha \tau \mathrm{d}\tau \quad \gamma_\alpha = \int_0^1 G_\alpha \mathrm{d}\tau \Big/ \int_0^1 \dot{G}_\alpha \mathrm{d}\tau; \ \alpha = 1,2 \tag{4}$$

Clearly, many known and new procedures can be developed assuming different values for β_α and γ_α. For example,

$$\gamma_1 = -\frac{1}{2}; \ \gamma_2 = \frac{1}{2}; \ \beta_1 = -\frac{1}{6}; \ \beta_2 = \frac{1}{3} \tag{5}$$

will produce the linear acceleration method obtained in the Comment, and the reader is referred to Ref. 3 for more details.

However, the authors believe that one does not fully exploit the power of the method by reproducing well-known existing operators, as for example in the Comment and these were not the stated objectives of the paper.[1] The purpose of the paper was to show that the finite element method based on the variational statement of Hamilton's Law[6] and applied in the time domain is capable of systematic derivation of many new highly accurate procedures for the solution of initial value problems.

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